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ARITHMETIC PROPERTIES OF PERIODIC MAPS

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Abstract. Let $\psi_1,\ldots,\psi_k$ be periodic maps from $\mathbb{Z}$ to a field of characteristic p (where p is zero or a prime). Assume that positive integers $n_1,\ldots,n_k$ not divisible by p are their periods respectively. We show that $\psi_1+\cdots+\psi_k$ is constant if $\psi_1(x)+\cdots+\psi_k(x)$ equals a constant for $|S|$ consecutive integers x where $S=\bigcup_{s=1}^k\{r/n_s:\ r=0,\ldots,n_s-1\}$. We also present some new results on finite systems of arithmetic sequences.

1. Introduction

For $a\in\mathbb{Z}$ and $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ we call

$$a(n)=a+n\mathbb{Z}=\{a+nx:\ x\in\mathbb{Z}\}$$

an *arithmetic sequence* with modulus n. For a finite system

$$A=\{a_s(n_s)\}_{s=1}^k \tag{1.1}$$

of such sequences, the *covering function* $w_A\colon\mathbb{Z}\to\mathbb{Z}$ given by

$$w_A(x)=|\{1\leqslant s\leqslant k:\ x\in a_s(n_s)\}| \tag{1.2}$$

is obviously periodic modulo the least common multiple $[n_1,\ldots,n_k]$ of all the moduli $n_1,\ldots,n_k$. If $w_A(x)\leqslant 1$ for all $x\in\mathbb{Z}$ (i.e., $a_i(n_i)\cap a_j(n_j)=\emptyset$ if $1\leqslant i<j\leqslant k$), then we say that (1.1) is *disjoint*. When $w_A(x)\geqslant 1$ for all $x\in\mathbb{Z}$ (i.e., $\bigcup_{s=1}^k a_s(n_s)=\mathbb{Z}$), (1.1) is called a *cover* of $\mathbb{Z}$.

A famous result of H. Davenport, L. Mirsky, D. Newman and R. Radó (cf. [NZ]) states that if (1.1) is a disjoint cover of $\mathbb{Z}$ with $1<n_1\leqslant\cdots\leqslant n_{k-1}\leqslant n_k$ then we must have $n_{k-1}=n_k$. In 1958 S. K. Stein [St] conjectured that if (1.1) is disjoint with $1<n_1<\cdots<n_k$ then there exists an integer $x\notin\bigcup_{s=1}^k a_s(n_s)$ with $1\leqslant x\leqslant 2^k$. In 1965 P. Erdős [E2] offered a prize for a proof of his following stronger conjecture (see [E1]):

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(1.1) forms a cover of $\mathbb{Z}$ if it covers those integers from 1 to 2^k. (The above 2^k is best possible because $\{2^{s-1}(2^s)\}_{s=1}^k$ covers $1, \ldots, 2^k - 1$ but does not cover any multiple of 2^k.) In 1969–1970 R. B. Crittenden and C. L. Vanden Eynden [CV1, CV2] supplied a long and awkward proof of the Erdős conjecture for $k \geqslant 20$.

Let m be a positive integer. In [Su4, Su5] the author called (1.1) an *m-cover* of $\mathbb{Z}$ if $w_A(x) \geqslant m$ for all $x \in \mathbb{Z}$, and an *exact m-cover* of $\mathbb{Z}$ if $w_A(x) = m$ for all $x \in \mathbb{Z}$. Recently the author [Su10] found that m-covers of $\mathbb{Z}$ are closely related to subset sums in a field and zero-sum problems on abelian groups.

Here is a result of [Su4, Su5] stronger than Erdős' conjecture: (1.1) *forms an m-cover of $\mathbb{Z}$ if it covers $|\{\{\sum_{s\in I} m_s/n_s\} : I \subseteq \{1, \ldots, k\}\}|$ consecutive integers at least m times, where the given $m_1, \ldots, m_k \in \mathbb{Z}^+$ are relatively prime to $n_1, \ldots, n_k$ respectively.* (As usual the fractional part of a real number x is denoted by $\{x\}$.) In [Su5] the author asked whether we have a similar result for exact m-covers of $\mathbb{Z}$. The answer is actually negative, moreover there is no constant $c(k,m) \in \mathbb{Z}^+$ such that (1.1) forms an exact m-cover of $\mathbb{Z}$ whenever it covers $c(k,m)$ consecutive integers exactly m times. In fact, if (1.1) is an exact m-cover of $\mathbb{Z}$ then for any integer $N > 1$ the system $\{a_1(n_1), \ldots, a_k(n_k), 0(N)\}$ covers $1, \ldots, N-1$ exactly m times but covers 0 exactly $m+1$ times! (This observation is due to the author's student H. Pan.)

For an assertion P we adopt Iverson's notation

$$[\![P]\!] = \begin{cases} 1 & \text{if } P \text{ holds,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

Observe that $w_A(x) = \sum_{s=1}^k \psi_s(x)$ where $\psi_s(x) = [\![n_s \mid x - a_s]\!]$ is periodic modulo n_s.

Our first result is completely new!

Theorem 1.1. *Let F be a field of characteristic p where p is zero or a prime. Let $n_1, \ldots, n_k$ be positive integers not divisible by p, and let $\psi_1, \ldots, \psi_k$ be maps from $\mathbb{Z}$ to F with periods $n_1, \ldots, n_k$ respectively. Then $\psi_1 + \cdots + \psi_k = 0$ if $\psi_1(x) + \cdots + \psi_k(x) = 0$ for $\sum_{d\in D} \varphi(d)$ consecutive integers x, where φ is Euler's totient function, $D = \bigcup_{s=1}^k D(n_s)$, and $D(n)$ denotes the set of positive divisors of $n \in \mathbb{Z}^+$.*

Remark 1.1. Clearly $\sum_{d\in D} \varphi(d)$ in Theorem 1.1 equals the cardinality of the set

$$\bigcup_{d\in D} \left\{\frac{c}{d} :\ 0 \leqslant c < d \text{ and } (c,d) = 1\right\} = \bigcup_{s=1}^k \left\{\frac{r}{n_s} :\ r = 0, 1, \ldots, n_s - 1\right\},$$

where (c,d) is the greatest common divisor of c and d. The result stated in the abstract is equivalent to Theorem 1.1 since a constant can be viewed as a function on $\mathbb{Z}$ periodic mod 1.

Corollary 1.1. *Let $w(x)$ be a function from $\mathbb{Z}$ to $\mathbb{Z}$ with period $n_0 \in \mathbb{Z}^+$. Then $w(x)$ is the covering function of* (1.1) *if $w_A(x) = w(x)$ for $|\bigcup_{s=0}^k \{0, 1/n_s, \ldots, (n_s-1)/n_s\}| \leqslant n_0 + n_1 + \cdots + n_k - k$ consecutive integers x. In particular,* (1.1) *forms an exact m-cover of $\mathbb{Z}$ if it covers $|\bigcup_{s=1}^k \{r/n_s :\ r = 0, \ldots, n_s - 1\}|$ consecutive integers exactly m times.*

Proof. Let $D = \bigcup_{s=0}^k D(n_s)$. As

$$\psi(x) := w_A(x) - w(x) = -w(x) + \sum_{s=1}^k [\![n_s \mid x - a_s]\!]$$

vanishes for $|\bigcup_{s=0}^k \{r/n_s :\ r = 0, \ldots, n_s - 1\}| = \sum_{d \in D} \varphi(d)$ consecutive integers x, we have $\psi(x) = 0$ for all $x \in \mathbb{Z}$ by Theorem 1.1. When $n_0 = 1$ and $w(x) = m \in \mathbb{Z}^+$, this yields the latter result in Corollary 1.1. $\square$

Remark 1.2. The problem whether a given $A = \{a_s(n_s)\}_{s=1}^k$ forms a cover of $\mathbb{Z}$ is known to be co-NP-complete. (See, e.g. [GJ] and [T].) However, Corollary 1.1 indicates that we can check whether system A has a given covering function in polynomial time! In 1997 the author [Su6] showed that if (1.1) covers all the integers the same number of times then

$$\left\{ \sum_{s \in I} \frac{1}{n_s} :\ I \subseteq \{1, \ldots, k\} \right\} \supseteq \bigcup_{s=1}^k \left\{ \frac{r}{n_s} :\ r = 0, \ldots, n_s - 1 \right\}.$$

Example 1.1. Let (1.1) be an exact m-cover of $\mathbb{Z}$, and let n be an integer greater than n_k. Then the system

$$A' = \{a_1(n_1), \ldots, a_{k-1}(n_{k-1}), a_k + n_k(n)\}$$

covers each of the consecutive integers $a_k + 1, \ldots, a_k + 2n_k - 1$ exactly m times but it does not cover a_k or $a_k + 2n_k$ exactly m times. For example, $B = \{1(2), 2(4), 0(4)\}$ is a disjoint cover of $\mathbb{Z}$, thus $B' = \{1(2), 2(4), 4(6)\}$ covers $1, \ldots, 7$ exactly once but it is not a disjoint cover. Note that the set $\bigcup_{n \in \{2,4,6\}} \{r/n :\ r = 0, \ldots, n-1\}$ just has 8 elements.

Corollary 1.2. *Let* (1.1) *be a system of arithmetic sequences, and let m be any integer greater than $k - f([n_1, \ldots, n_k])$.* (*The function f is given by $f(1) = 0$ and $f(\prod_{i=1}^r p_i) = \sum_{i=1}^r (p_i - 1)$ where $p_1, \ldots, p_r$ are primes.*) *Then there is an $x \in \{0, 1, \ldots, |S| - 1\}$ such that $w_A(x) \neq m$ where $S = \bigcup_{s=1}^k \{r/n_s :\ r = 0, 1, \ldots, n_s - 1\}$.*

Proof. If (1.1) is an exact m-cover of $\mathbb{Z}$, then $k \geqslant m + f([n_1, \ldots, n_k])$ by Corollary 4.5 of [Su7]. Thus, in view of the condition, (1.1) does not form an exact m-cover of $\mathbb{Z}$ and hence the desired result follows from Corollary 1.1. $\square$

Our next theorem extends some earlier work in [Su4, Su5].

Theorem 1.2. *Let $n_1, \dots, n_k$ be positive integers, and let $R_1, \dots, R_k$ be finite subsets of $\mathbb{Z}$. For $s = 1, \dots, k$, let c_{st} lie in the complex field $\mathbb{C}$ for each $t \in R_s$, and set*

$$X_s = \Big\{ x \in \mathbb{Z} : \sum_{t \in R_s} c_{st} e^{2\pi i \frac{t}{n_s} x} = 0 \Big\}. \tag{1.4}$$

If the system $\{X_s\}_{s=1}^k$ covers W consecutive integers at least m times where $1 \leqslant m \leqslant k$ and

$$W = \max_{\substack{I \subseteq \{1,\dots,k\} \\ |I| = k-m+1}} \left| \left\{ \left\{ \sum_{s \in I} \frac{r_s}{n_s} \right\} : \ r_s \in R_s \right\} \right| \leqslant \max_{\substack{I \subseteq \{1,\dots,k\} \\ |I| = k-m+1}} \prod_{s \in I} |R_s|, \tag{1.5}$$

then it covers every integer at least m times.

Corollary 1.3. *Let* (1.1) *be a system of arithmetic sequences, and let $m_1, \dots, m_k$ be integers relatively prime to $n_1, \dots, n_k$ respectively. Let l be any nonnegative integer with $w_A(x) \geqslant l$ for all $x \in \mathbb{Z}$, and set*

$$W_l = \max_{\substack{I \subseteq \{1,\dots,k\} \\ |I| = k-l}} \left| \left\{ \left\{ \sum_{s \in J} \frac{m_s}{n_s} \right\} : \ J \subseteq I \right\} \right| \leqslant 2^{k-l}. \tag{1.6}$$

Then the covering function $w_A(x)$ takes its minimum on every set of W_l consecutive integers.

Proof. Without loss of generality we may assume that $1 \leqslant m_s \leqslant n_s$ for all $s = 1, \dots, k$. As $m(A) = \min_{x \in \mathbb{Z}} w_A(x) \geqslant l$ and $W_l \geqslant W_{m(A)}$, it suffices to work with $l = m(A)$ below.

The case $l = k$ is trivial, so we let $l < k$. Set $c_{s0} = 1$ and $c_{sm_s} = -e^{-2\pi i a_s m_s / n_s}$ for $s = 1, \dots, k$. Since m_s and n_s are relatively prime,

$$X_s := \Big\{ x \in \mathbb{Z} : \ c_{s0} e^{2\pi i \frac{0}{n_s} x} + c_{sm_s} e^{2\pi i \frac{m_s}{n_s} x} = 0 \Big\} = a_s(n_s).$$

Applying Theorem 1.2 with $m = l + 1$ and $R_s = \{0, m_s\}$ $(1 \leqslant s \leqslant k)$, we immediately get the desired result. $\square$

Remark 1.3. (a) [Su9] contains some other interesting results on the covering function of (1.1). (b) W_l in (1.6) might be smaller than its value in the case $m_1 = \dots = m_k = 1$. Let $n_1 = 3,\ n_2 = 5$ and $n_3 = 15$. Set

$$W_0(m_1, m_2, m_3) = \left| \left\{ \left\{ \sum_{s \in J} \frac{m_s}{n_s} \right\} : \ J \subseteq \{1, 2, 3\} \right\} \right|$$

for $m_1, m_2, m_3 \in \mathbb{Z}$. Then $W_0(1, 1, 2) = 7 < W_0(1, 1, 1) = 8$.

Our third theorem characterizes the least period of a covering function.

Theorem 1.3. *Let $\lambda_s \in \mathbb{C}$, $a_s \in \mathbb{Z}$ and $n_s \in \mathbb{Z}^+$ for $s = 1, \ldots, k$. Then the smallest positive period n_0 of the (weighted) covering function*

$$w(x) = \sum_{s=1}^{k} \lambda_s [\![n_s \mid x - a_s]\!]$$

is the least $n \in \mathbb{Z}^+$ such that $\alpha n \in \mathbb{Z}$ for all those $\alpha \in [0, 1)$ with $\sum_{\substack{1 \leqslant s \leqslant k \\ \alpha n_s \in \mathbb{Z}}} \lambda_s n_s^{-1} e^{2\pi i \alpha a_s} \neq 0$.

Remark 1.4. Under the condition of Theorem 1.3, it can be easily checked that $\sum_{x=0}^{N-1} w(x)/N = \sum_{s=1}^{k} \lambda_s / n_s$ where $N = [n_1, \ldots, n_k]$. If $w(x) = 0$ for all $x \in \mathbb{Z}$, then $n_0 = 1$ and hence

$$\sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} = 0 \quad \text{for all } \alpha \in [0, 1). \tag{1.7}$$

This was first obtained by the author [Su2] in 1991 via an analytic method, and the converse was proved in [Su3]. In [Su8] the author determined those functions $f\colon \bigcup_{n \in \mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ such that $\sum_{s=1}^{k} \lambda_s f(a_s + n_s \mathbb{Z})$ only depends on the covering function $w(x)$, this was announced by the author [Su1] in 1989.

Let l be a positive integer, and let

$$\mathbb{Z}^l = \{\vec{x} = \langle x_1, \ldots, x_l \rangle : \ x_1, \ldots, x_l \in \mathbb{Z}\}$$

be the direct sum of l copies of the ring $\mathbb{Z}$. For $\vec{x}, \vec{y} \in \mathbb{Z}^l$, we use $\vec{x} \mid \vec{y}$ to mean that $\vec{y} = \vec{q}\vec{x} = \langle q_1 x_1, \ldots, q_l x_l \rangle$ for some $\vec{q} \in \mathbb{Z}^l$. A function $\Psi : \mathbb{Z}^l \to \mathbb{C}$ is said to be *periodic modulo* $\vec{n} \in \mathbb{Z}^l$ if $\Psi(\vec{x}) = \Psi(\vec{y})$ whenever $\vec{x} - \vec{y} = \langle x_1 - y_1, \ldots, x_l - y_l \rangle$ is divisible by $\vec{n}$. For $x_1, \ldots, x_l \in \mathbb{Z}$, we also use $[x_t]_{1 \leqslant t \leqslant l}$ to denote the least common multiple of $x_1, \ldots, x_l$.

Theorem 1.4. *Let $\lambda_s \in \mathbb{C}$, $\vec{a}_s \in \mathbb{Z}^l$ and $\vec{n}_s \in (\mathbb{Z}^+)^l$ for $s = 1, \ldots, k$ where $l \in \mathbb{Z}^+$. Suppose that the function*

$$w(\vec{x}) = \sum_{s=1}^{k} \lambda_s [\![\vec{n}_s \mid \vec{x} - \vec{a}_s]\!] \tag{1.8}$$

is periodic modulo $\vec{n}_0 \in (\mathbb{Z}^+)^l$. Let $\vec{d} \in (\mathbb{Z}^+)^l$, $\vec{d} \nmid \vec{n}_0$ and

$$I(\vec{d}) = \{1 \leqslant s \leqslant k : \ \vec{d} \mid \vec{n}_s\} \neq \emptyset.$$

If $\sum_{s\in I(\vec{d}\,)}\lambda_s/(n_{s1}\cdots n_{sl})\neq 0$, *then*

$$|I(\vec{d}\,)|\geqslant\left|\left\{\left\{\sum_{t=1}^{l}\frac{a_{st}}{d_t}\right\}: s\in I(\vec{d}\,)\right\}\right|\geqslant \min_{\substack{0\leqslant s\leqslant k\\ \vec{d}\nmid\vec{n}_s}}\left[\frac{d_t}{(d_t,n_{st})}\right]_{1\leqslant t\leqslant l}\geqslant p(d_1\cdots d_l)$$

where we use $p(m)$ *to denote the least prime divisor of an integer* $m>1$.

Remark 1.5. Theorem 1.4 is a generalization of the main result of [Su2] which corresponds to the case $l=1$ and improves the Znám–Newman result [N].

Corollary 1.4. *Let* $\lambda_s\in\mathbb{C}\setminus\{0\}$, $\vec{a}_s\in\mathbb{Z}^l$ *and* $\vec{n}_s\in(\mathbb{Z}^+)^l$ *for* $s=1,\dots,k$ *where* $l\in\mathbb{Z}^+$. *Suppose that all those moduli* $\vec{n}_s$ *which are maximal with respect to divisibility are distinct. Then the function* $w(\vec{x})$ *given by* (1.8) *is periodic modulo* $\vec{n}_0\in(\mathbb{Z}^+)^l$ *if and only if* $\vec{n}_0$ *is divisible by all the moduli* $\vec{n}_1,\dots,\vec{n}_k$.

Proof. If $\vec{n}_s\mid\vec{n}_0$ for all $s=1,\dots,k$, then the function $w(\vec{x})$ is obviously periodic mod $\vec{n}_0$.

Now suppose that $w(\vec{x})$ is periodic modulo $\vec{n}_0$ but not all the moduli divide $\vec{n}_0$. Then there exists a maximal modulus $\vec{n}_r$ with respect to divisibility such that $\vec{n}_r\nmid\vec{n}_0$. By the condition,

$$I(\vec{n}_r):=\{1\leqslant s\leqslant k:\ \vec{n}_r\mid\vec{n}_s\}=\{1\leqslant s\leqslant k:\ \vec{n}_s=\vec{n}_r\}=\{r\}.$$

On the other hand, by Theorem 1.4 we should have $|I(\vec{n}_r)|\geqslant p(n_{r1}\cdots n_{rl})$. The contradiction ends our proof. □

Remark 1.6. Corollary 1.4 in the case $l=1$ was essentially established by Š. Porubský [P].

2. Proofs of Theorems 1.1–1.4

Lemma 2.1. *Let* $c_1,\dots,c_n$ *lie in a field* F, *and let* $z_1,\dots,z_n$ *be distinct elements of* $F\setminus\{0\}$. *If* $\sum_{j=1}^n c_jz_j^x$ *vanishes for* n *consecutive integers* x, *then it vanishes for all* $x\in\mathbb{Z}$.

Proof. Suppose that $\sum_{j=1}^n c_jz_j^{h+i-1}=0$ for every $i=1,\dots,n$ where $h\in\mathbb{Z}$. Since the Vandermonde determinant

$$\|z_j^{i-1}\|_{1\leqslant i,j\leqslant n}=\begin{vmatrix}1&1&\dots&1\\ z_1&z_2&\dots&z_n\\ \vdots&\vdots&\ddots&\vdots\\ z_1^{n-1}&z_2^{n-1}&\dots&z_n^{n-1}\end{vmatrix}=\prod_{1\leqslant i<j\leqslant n}(z_j-z_i)$$

does not vanish, by Cramer's rule we have $c_j z_j^h = 0$ and hence $c_j = 0$ for all $j = 1, \dots, n$. Therefore $\sum_{j=1}^n c_j z_j^x = 0$ for any $x \in \mathbb{Z}$. $\square$

Proof of Theorem 1.1. As p does not divide $N = [n_1, \dots, n_k]$, the equation $x^N - 1 = 0$ has N distinct roots in the algebraic closure E of the field F. The multiplicative group $\{\zeta \in E : \zeta^N = 1\}$ of order N is cyclic, so E contains an element ζ of multiplicative order N. For $a \in \mathbb{Z}$ and $1 \leqslant s \leqslant k$, we have the geometric series

$$\frac{1}{n_s} \sum_{r=0}^{n_s-1} \zeta^{\frac{N}{n_s} ar} = [\![n_s \mid a]\!]. \tag{2.1}$$

Therefore

$$\begin{aligned}
\sum_{s=1}^k \psi_s(x) &= \sum_{s=1}^k \sum_{a=0}^{n_s-1} [\![n_s \mid a - x]\!] \psi_s(a) \\
&= \sum_{s=1}^k \sum_{a=0}^{n_s-1} \frac{1}{n_s} \sum_{r=0}^{n_s-1} \zeta^{\frac{N}{n_s}(a-x)r} \psi_s(a) \\
&= \sum_{s=1}^k \frac{1}{n_s} \sum_{a=0}^{n_s-1} \psi_s(a) \sum_{\substack{0 \leqslant \alpha < 1 \\ \alpha n_s \in \mathbb{Z}}} \zeta^{\alpha N(a-x)} \\
&= \sum_{\alpha \in S} \left(\zeta^{-\alpha N}\right)^x \left(\sum_{s=1}^k \frac{[\![\alpha n_s \in \mathbb{Z}]\!]}{n_s} \sum_{a=0}^{n_s-1} \psi_s(a) \zeta^{\alpha N a} \right),
\end{aligned}$$

where S is the set

$$\{\alpha \in [0,1) : \alpha n_s \in \mathbb{Z} \text{ for some } 1 \leqslant s \leqslant k\} = \bigcup_{s=1}^k \left\{ \frac{r}{n_s} : r = 0, \dots, n_s - 1 \right\}.$$

As those $\zeta^{-\alpha N}$ with $\alpha \in S$ are distinct, applying Lemma 2.1 we find that $\sum_{s=1}^k \psi_s(x) = 0$ for $|S|$ consecutive integers x if and only if $\sum_{s=1}^k \psi_s(x) = 0$ for all $x \in \mathbb{Z}$. By Remark 1.1, $|S| = \sum_{d \in D} \varphi(d)$. This concludes the proof. $\square$

Proof of Theorem 1.2. Clearly an integer x is covered by $\{X_s\}_{s=1}^k$ at least m times if and only if x is covered by $\{X_s\}_{s \in I}$ for all $I \subseteq \{1, \dots, k\}$ with $|I| = k - m + 1$.

Now let $I \subseteq \{1, \dots, k\}$ and $|I| = k - m + 1$. For any $x \in \mathbb{Z}$, we have

$$\begin{aligned}
\prod_{s \in I} \sum_{t \in R_s} c_{st} e^{2\pi i \frac{t}{n_s} x} &= \sum_{r_s \in R_s \text{ for } s \in I} \left(\prod_{s \in I} c_{s r_s} \right) e^{2\pi i x \sum_{s \in I} r_s / n_s} \\
&= \sum_{\theta \in R(I)} C_{I,\theta} e^{2\pi i \theta x}
\end{aligned}$$

where

$$R(I)=\left\{\left\{\sum_{s\in I}\frac{r_s}{n_s}\right\}:\ r_s\in R_s\right\}\quad\text{and}\quad C_{I,\theta}=\sum_{\substack{r_s\in R_s\ \text{for}\ s\in I\\ \{\sum_{s\in I}r_s/n_s\}=\theta}}\prod_{s\in I}c_{sr_s}.$$

Since those $e^{2\pi i\theta}$ with $\theta\in R(I)$ are distinct, by Lemma 2.1 the system $\{X_s\}_{s\in I}$ covers $|R(I)|$ consecutive integers x if and only if it covers all $x\in\mathbb{Z}$.

In view of the above, we immediately obtain the desired result. □

Proof of Theorem 1.3. Let $S=\{0\leqslant\alpha<1:\ \alpha n_s\in\mathbb{Z}\text{ for some }1\leqslant s\leqslant k\}$ and

$$T=\left\{0\leqslant\alpha<1:\ c_\alpha=\sum_{\substack{1\leqslant s\leqslant k\\ \alpha n_s\in\mathbb{Z}}}\frac{\lambda_s}{n_s}e^{2\pi i\alpha a_s}\neq 0\right\}.$$

For each $s=1,\ldots,k$ the arithmetical function $\psi_s(x)=\lambda_s[\![n_s\mid x-a_s]\!]$ is periodic modulo n_s. By the proof of Theorem 1.1, for any $x\in\mathbb{Z}$ we have

$$w(x)=\sum_{s=1}^k\lambda_s[\![n_s\mid x-a_s]\!]=\sum_{\alpha\in S}e^{-2\pi i\alpha x}c_\alpha=\sum_{\alpha\in T}e^{-2\pi i\alpha x}c_\alpha.$$

Let n be the least positive integer such that $\alpha n\in\mathbb{Z}$ for all $\alpha\in T$. By the above, $w(x)=w(x+n)$ for all $x\in\mathbb{Z}$. Thus $n_0\mid n$.

If $T=\emptyset$, then $n=1$ and hence $n_0=n$. In the case $T\neq\emptyset$, we have

$$0=w(x)-w(x+n_0)=\sum_{\alpha\in T}e^{-2\pi i\alpha x}(1-e^{-2\pi i\alpha n_0})c_\alpha$$

for every $x=0,\ldots,|T|-1$, and hence $(1-e^{-2\pi i\alpha n_0})c_\alpha=0$ for any $\alpha\in T$ (Vandermonde). Now that $\alpha n_0\in\mathbb{Z}$ (i.e., $e^{-2\pi i\alpha n_0}=1$) for all $\alpha\in T$, we have $n_0\geqslant n$ and thus $n_0=n$.

The proof of Theorem 1.3 is now complete. □

Proof of Theorem 1.4. Let $\vec{c}$ be any vector in $\mathbb{Z}^l$ with $\vec{d}\nmid\vec{c}\vec{n}_0$. Then, for some $1\leqslant r\leqslant l$ we have $d_r\nmid c_rn_{0r}$ Note that $\vec{n}_0$ divides the vector $\langle 0,\ldots,0,n_{0r},0,\ldots,0\rangle$. For any $x_1,\ldots,x_{r-1},x_{r+1},\ldots,x_l\in\mathbb{Z}$, since

$$\sum_{s=1}^k\left(\lambda_s\prod_{\substack{t=1\\ t\neq r}}^l[\![n_{st}\mid x_t-a_{st}]\!]\right)[\![n_{sr}\mid x_r-a_{sr}]\!]=w(\vec{x})$$

is periodic mod n_{0r} as a function of x_r, by Theorem 1.3 we must have

$$\sum_{\substack{s=1\\ d_r\mid c_rn_{sr}}}^k\left(\lambda_s\prod_{\substack{t=1\\ t\neq r}}^l[\![n_{st}\mid x_t-a_{st}]\!]\right)\frac{e^{2\pi i(c_r/d_r)a_{sr}}}{n_{sr}}=0.$$

(Recall that $(c_r/d_r)n_{0r} \notin \mathbb{Z}$.)

Let $J = \{1 \leqslant s \leqslant k: \ d_r \mid c_r n_{sr}\}$ and $\lambda'_s = \lambda_s n_{sr}^{-1} e^{2\pi i a_{sr} c_r/d_r}$ for $s \in J$. Given $r' \in \{1, \dots, l\} \setminus \{r\}$ and $x_t \in \mathbb{Z}$ with $t \neq r, r'$, we have

$$\sum_{s\in J}\Bigg(\lambda'_s \prod_{\substack{t=1\\ t\neq r,r'}}^{l} [\![n_{st} \mid x_t - a_{st}]\!]\Bigg)[\![n_{sr'} \mid x_{r'} - a_{sr'}]\!]$$
$$= \sum_{s\in J}\lambda'_s \prod_{\substack{t=1\\ t\neq r}}^{l}[\![n_{st} \mid x_t - a_{st}]\!] = 0$$

for all $x_{r'} \in \mathbb{Z}$. By applying Remark 1.4 $l-1$ times we finally obtain that

$$\sum_{\substack{s=1\\ \vec{d}\mid \vec{c}\vec{n}_s}}^{k} \frac{\lambda_s}{n_{s1}\cdots n_{sl}} e^{2\pi i \sum_{t=1}^{l} a_{st}c_t/d_t} = 0. \tag{2.2}$$

Set $m = \min_{0\leqslant s\leqslant k,\, \vec{d}\nmid \vec{n}_s}[d_t/(d_t, n_{st})]_{1\leqslant t\leqslant l}$. Clearly $m \geqslant p(d_1\cdots d_l)$. Let c be any positive integer less than m. For $s = 0, 1\cdots, k$ we have

$$\vec{d} \mid c\vec{n}_s \Leftrightarrow d_t \mid cn_{st} \text{ for all } t = 1, \dots, l \Leftrightarrow \left[\frac{d_t}{(d_t, n_{st})}\right]_{1\leqslant t\leqslant l} \Bigg| \ c \Leftrightarrow \vec{d} \mid \vec{n}_s.$$

In other words, $\vec{d} \mid c\vec{n}_s$ if and only if $s \in I(\vec{d})$. (2.2) in the case $\vec{c} = \langle c, \dots, c\rangle$ yields that

$$\sum_{s\in I(\vec{d})} \frac{\lambda_s}{n_{s1}\cdots n_{sl}} e^{2\pi i c\sum_{t=1}^{l} a_{st}/d_t} = 0.$$

Let $\Theta = \{\{\sum_{t=1}^{l} a_{st}/d_t\} : \ s \in I(\vec{d})\}$. Suppose that $|\Theta| < m$. Then for each $c = 1, \dots, |\Theta|$ we have

$$\sum_{\theta\in\Theta} e^{2\pi i c\theta} \sum_{\substack{s\in I(\vec{d})\\ \{\sum_{t=1}^{l} a_{st}/d_t\}=\theta}} \frac{\lambda_s}{n_{s1}\cdots n_{sl}}$$
$$= \sum_{s\in I(\vec{d})} \frac{\lambda_s}{n_{s1}\cdots n_{sl}} e^{2\pi i c\sum_{t=1}^{l} a_{st}/d_t} = 0.$$

By Lemma 2.1 this holds for all integers c, in particular $c = 0$:

$$\sum_{s\in I(\vec{d})} \frac{\lambda_s}{n_{s1}\cdots n_{sl}} = 0.$$

This directly contradicts one of the hypotheses, whence $|\Theta| \geqslant m$. $\square$

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